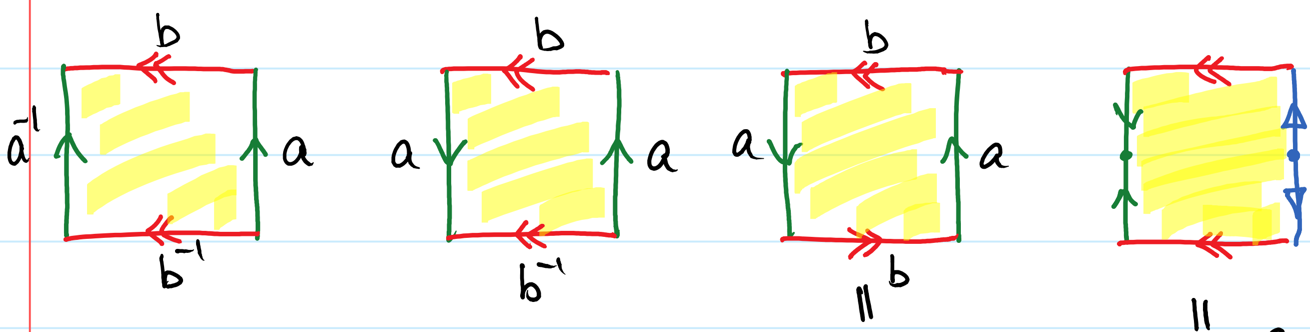
March 9, 2017

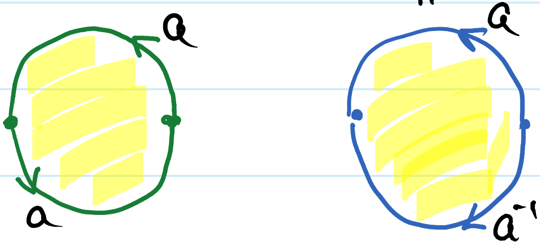
Friday, March 10, 2017 10:16 AM

Recall several quotients on $[0,1]\times[0,1]$

Torus | Klein Bottle | Projective Plane | Sphere S^2

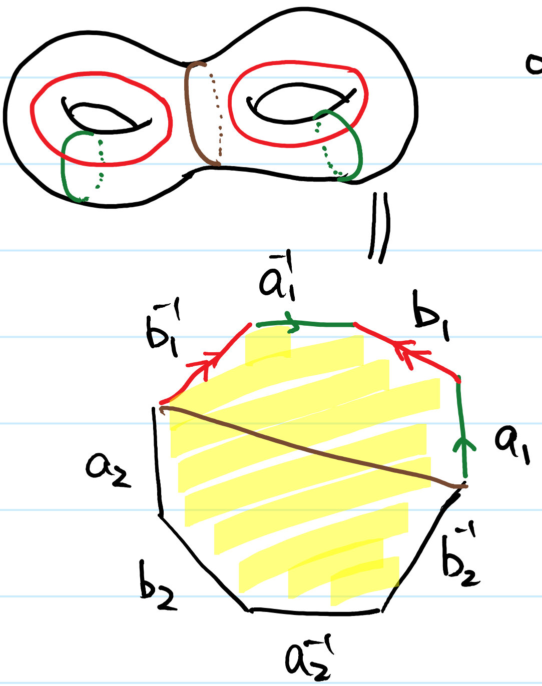


Let us denote the sides of $[0,1]\times[0,1]$ by symbols in acord with the "gluing direction"

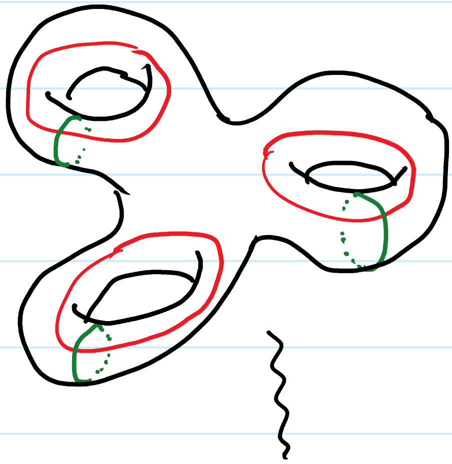


How to get other surface?

For example,



or



$a_1, b_1, a_2, b_2, a_3, b_3$

Classification of Surfaces

A compact surface without boundary is homeomorphic to one below.

(1) Sphere S^2

(2) $(4n\text{-gon})/\sim$ where $\sim$ is determined by the condition $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$

(3) $(2n\text{-gon})/\sim$ where $\sim$ is determined by the condition $a_1^2 a_2^2 \cdots a_n^2$.

Understanding the statement.

Boundary exists. D^2, $S^1 \times [0,1]$, etc.

Non-compact. $\mathbb{R}^2$, $S^1 \times (0,1)$, etc.

Type (1)+(2), in some books, are called

(a) compact orientable surfaces

(b) closed surfaces in $\mathbb{R}^3$

Remarks.

* There are other formulations, e.g., handle body
* The formulation with complex structure is related to their universal covering space.
* The brown curve in the double torus is a process denoted by  $= \mathbb{T} \# \mathbb{T}$. We have relation such as $\mathbb{T} \# \mathbb{P} = \mathbb{P} \# \mathbb{P} =$ Klein.

03-09-p3

Monday, March 13, 2017 2:04 PM

Definition

Given $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$. The **disjoint union**, $X \sqcup Y$, of them is the set $X \times \{0\} \cup Y \times \{1\}$ with the topology generated by

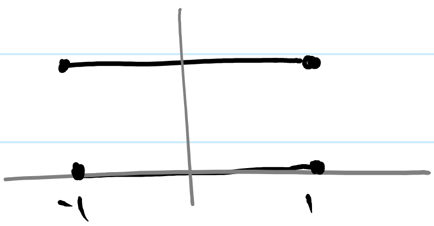
$$\{U \times \{0\} : U \in \mathcal{T}_X\} \cup \{V \times \{1\} : V \in \mathcal{T}_Y\}$$

Remark. This is basically putting two spaces "side-by-side".

Example. Let $X = Y = [-1, 1]$, standard topology

Then $X \cup Y = [-1, 1]$, while

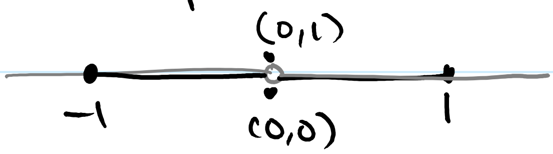
$X \sqcup Y =$



same as subspace in $\mathbb{R}^2$

Very Important Example. On $[-1,1] \sqcup [-1,1]$, identify $(x, 0)$ with $(x, 1)$ for $x \neq 0$

The illustrative picture is



This space is not Hausdorff

Every nbhd of $(0,1)$ and of $(0,0)$ will intersect at $(-\varepsilon, 0) \cup (0, \varepsilon)$ for some $\varepsilon > 0$.

03-09-p4

Monday, March 13, 2017 2:05 PM

Definition. Given X, Y, $f: A \subset X \longrightarrow Y$

$X \cup_f Y$ is the quotient space $(X \sqcup Y)/\sim$

where $\underbrace{a \in A \sim f(a) \in Y}$

in fact, $(a,0), (f(a),1) \in A \sqcup Y \subset X \sqcup Y$

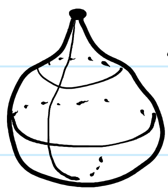
Language. It is attaching X to Y along $f: A \to Y$

Examples.

* $X = Y = \mathbb{D}^2$, $A = S^1 \subset \mathbb{D}^2$, $f: S^1 \longrightarrow Y$ is identity

  $X \cup_f Y = S^2$, in the form of gluing two disc as North-South-hemisphere

* $X = \mathbb{D}^2$, $A = S^1 \subset X$, $Y = \{y_0\}$, f is constant

  $X \cup_f Y =$  $= S^2$ again

Exercise. Let $X = Y = \mathbb{D}^2$, $A = S^1 \subset X$

Find $X \cup_f Y$ if

(a) $f: S^1 \longrightarrow \mathbb{D}^2 : f(e^{i\theta}) = e^{2i\theta}$

(b) $f: S^1 \longrightarrow \mathbb{D}^2 : f(e^{i\theta}) = e^{-i\theta}$

(c) $f: S^1 \longrightarrow \mathbb{D}^2 : f(e^{i\theta}) = -e^{i\theta}$